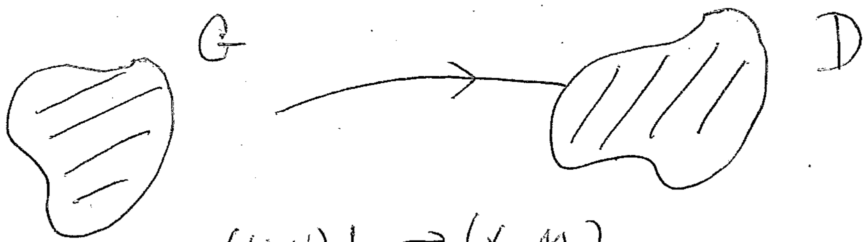
2020 B
Week 7 (Feb 23)

Consider a map from region G to another region D:



$$(u,v) \longmapsto (x,y)$$

$$x = g(u,v)$$
$$y = h(u,v)$$

A function f in D can be pull-back to become a function in G by

$$\hat{f}(u,v) = f(g(u,v), h(u,v)).$$

The change of variables formula: Suppose $(u,v) \mapsto (g,h)$ is 1-1 onto from the interior of G to the interior of D. Then for any piecewise continuous function f in D,

$$\iint_D f(x,y)\,dA(x,y) = \iint_G \hat{f}(u,v) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA(u,v)$$

where $\frac{\partial(x,y)}{\partial(u,v)}$ is the Jacobian determinant of (g,h).

e.g. Replacing u, v by r, θ, let

$$x = g(r,\theta) = r\cos\theta,$$
$$y = h(r,\theta) = r\sin\theta.$$

Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = \cos\theta \, r\cos\theta - (-r\sin\theta)\sin\theta$

$= r \geq 0$

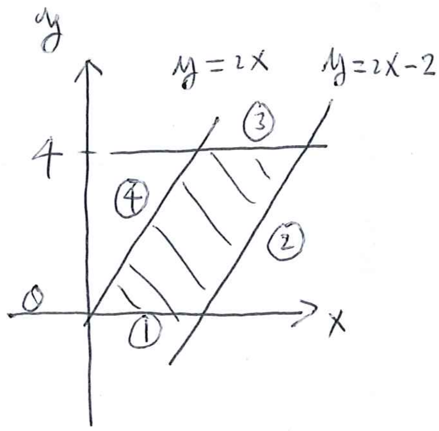
So, the formula is

$$\iint_D f(x,y)\,dA(x,y) = \iint_G \hat{f}(r,\theta)\, r\, dA(r,\theta)$$

When $D = \{(x,y) : r_1(\theta) \leq r \leq r_2(\theta),\ \theta_1 \leq \theta \leq \theta_2\}$, we recover

$$\iint_D f(x,y)\,dA(x,y) = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta \quad \#$$

e.g. Evaluate $\displaystyle\int_0^4 \int_{y/2}^{y/2+1} \frac{2x-y}{2}\,dx\,dy$.



We see that D is $y/2 \leq x \leq y/2 + 1$
$0 \leq y \leq 4$.

Let $u = \dfrac{2x-y}{2}$, $v = \dfrac{y}{2}$, ie,

$x = u + v$, $y = 2v$.

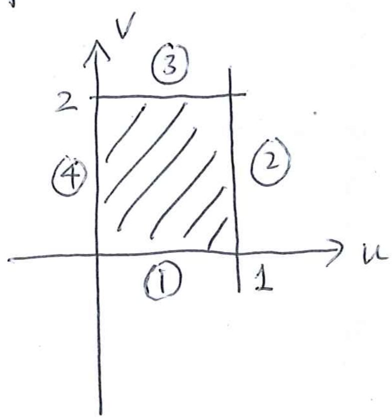
To find G we look at the boundary correspondence.

① $y = 0 \Longleftrightarrow u = x, v = 0$ ie, $v = 0$

② $y = 2x - 2 \Longleftrightarrow u = 1$

③ $y = 4 \Longleftrightarrow v = 2$

④ $y = 2x \Longleftrightarrow u = 0$



G is the rectangle $[0,1] \times [0,2]$

$$\therefore \int_0^4 \int_{y/2}^{y/2+1} \frac{2x-y}{2}\,dx\,dy = \iint_D \frac{2x-y}{2}\,dA(x,y)$$

$$= \iint_G u \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA(u,v)$$

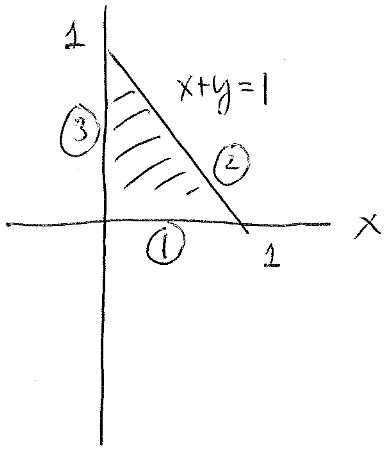
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2$$

$$= 2\iint_G u\,dA(u,v)$$

$$= 2\int_0^1\int_0^2 u\,dv\,du$$

$$= 2 \quad \#$$

e.g. Evaluate $\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx$



D is $0 \le y \le 1-x$
$\quad 0 \le x \le 1$

Let $u = x+y$
$\quad v = y-2x$, ie,

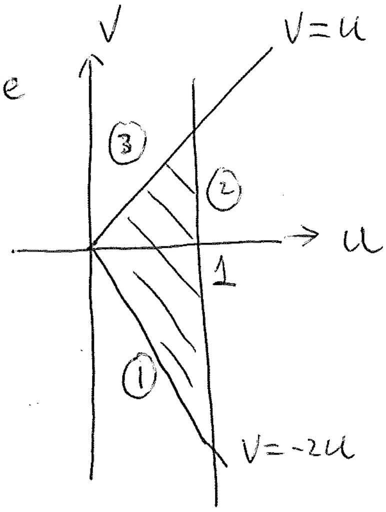
$$x = \frac{1}{3}(u-v), \quad y = \frac{2}{3}u + \frac{1}{3}v$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{vmatrix} = \frac{1}{9} - \left(-\frac{1}{3}\right)\frac{2}{3} = \frac{3}{9} = \frac{1}{3}.$$

$G = ?$ ① $y=0 \longleftrightarrow u=x,\ v=-2x$, ie
$\quad v=-2u$

② $x+y=1 \longleftrightarrow u=1$

③ $x=0 \longleftrightarrow u=y,\ v=y$, ie
$\quad u=v$



$$\therefore \int_0^1\int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx = \iint_D \sqrt{x+y}\,(y-2x)^2\,dA(x,y)$$

$$= \iint_G \sqrt{u}\,v^2\,\frac{1}{3}\,dA(u,v)$$

$$= \frac{1}{3}\int_0^1\int_{-2u}^{u} \sqrt{u}\,v^2\,dv\,du$$

$$\vdots$$

$$= 2/9 \quad \#$$

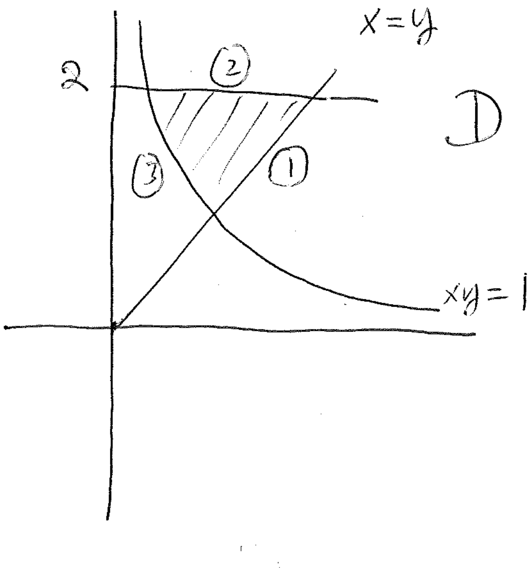
Summarizing the steps:

(I) Choose u, v in order to simplify the integrand $f(x,y)$ or D,

(II) To determin G by looking at the boundary correspondence.

(III) apply the change of variables formula.
(don't forget the absolute value of $\frac{\partial(x,y)}{\partial(u,v)}$!)

e.g. $\displaystyle\int_1^2\int_{1/y}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\,dx\,dy$



Let $v=\sqrt{\frac{y}{x}}$, $u=\sqrt{xy}$.

Then $v^2 = y/x$, $u^2 = xy$, so

$v^2u^2 = y/x \; xy = y^2$, $y = uv$

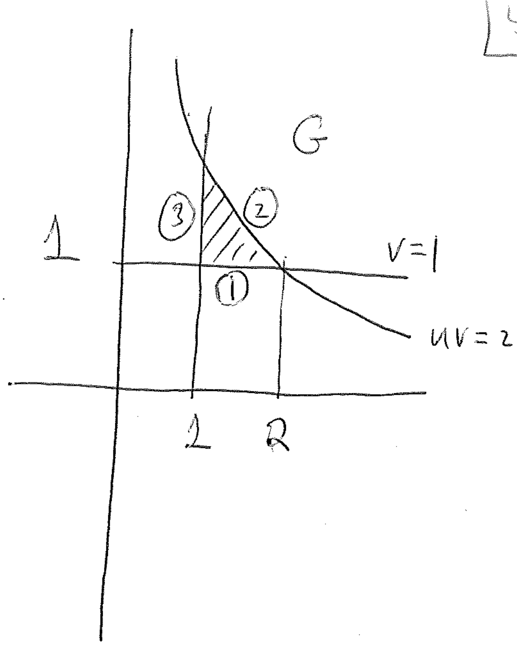
then $x = y/v^2 = uv/v^2 = u/v$.

$\therefore \quad \boxed{x = u/v,\; y = uv}$

① $x=y \longleftrightarrow \frac{u}{v}=uv$, ie $v=1$

② $y=2 \longleftrightarrow uv=2$

③ $xy=1 \longleftrightarrow u=1$



$\therefore \int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy$

$$= \iint_D \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dA(x,y)$$

$$= \iint_G v e^u \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA(u,v)$$

$$= \int_1^2 \int_1^{2/u} v e^u \frac{2u}{v}\, dv\, du$$

$$= 2\int_1^2 \int_1^{2/u} u e^u\, dv\, du$$

$\vdots$

$$= 2e(e-2) \quad \#$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix}$$

$$= \frac{u}{v} + \frac{u}{v}$$

$$= \frac{2u}{v}$$

• Ideas of Proof of the Change of Variables formula

• Principle I. Interior Riemann sums approximation.

As $\|P\| \to 0$, the interior Riemann sums

$$\to \iint_D f \, dA.$$

Explanation: Recall

$$\iint_D f \, dA \overset{\text{def}}{=} \iint_R \hat{f} \, dA,$$

$\hat{f}$ universal extension

$$\hat{f} = \begin{cases} f & \text{in } D \\ 0 & \text{outside } D \end{cases}$$

A partition P divide R into subrectgle R_{jk}. Let

$\mathcal{A}$ = those R_{jk} completely sitting inside R

$\mathcal{B}$ = those R_{jk} touching some parts of $R \setminus D$.

Interior Riemann sums

$$= \sum_{R_{jk} \in \mathcal{A}} f(P_{jk}) \Delta x_j \Delta y_k .$$

Proof of Principle I:

As $\|P\| \to 0$, $\displaystyle\sum_{j,k} \hat{f}(P_{jk}) \Delta x_j \Delta y_k \to \iint_D f \, dA = \iint_R \hat{f} \, dA$

when $R_{jk} \in \mathcal{B}$, choose $P_{jk} \notin D$, $\hat{f}(P_{jk}) = 0$.

when $R_{jk} \in \mathcal{A}$, $\hat{f}(P_{jk}) = f(P_{jk})$.

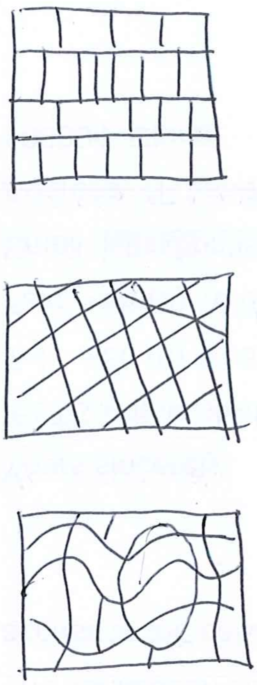
$$\therefore \sum_{j,k} \hat{f}(P_{jk}) \Delta x_j \Delta y_k = \sum_{\mathcal{A}} f(P_{jk}) \Delta x_j \Delta y_k \to \iint_D f \, dA, \text{ done.}$$

• Generalized Riemann Sums.

A generalized partition is $D = \bigcup_{j=1}^{N} P_j$, where

① each D_j is a region, and

② $D_j \cap D_k = \phi$ or $D_j \cap D_k$ is some curves.



generalized partitions of R

$$\|P\| = \max\{\text{diam } D_1, \dots, \text{diam } D_N\}$$

Generalized Riemann sum

$$R(f, P) = \sum_{j=1}^{N} f(p_j)|D_j|, \text{ where}$$

$p_j \in D_j$ and

$$|D_j| = \iint_{D_j} 1 \, dA.$$

We accept that, as $\|P\| \to 0$,

$$R(f, P) \to \iint_D f \, dA$$ when f is piecewise continuous.

Moreover, Principle I is still true for generalized Riemann sums, that is,

Principle II As $\|P\| \to 0$

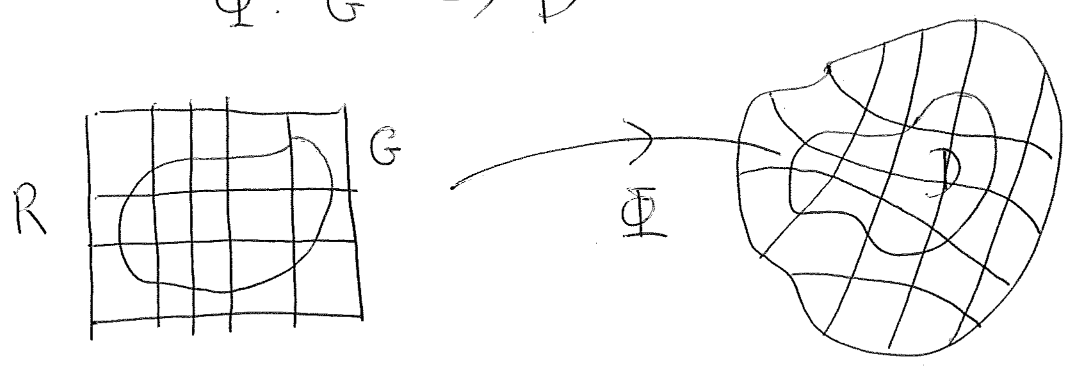
$$\sum_{D_j \in A} f(p_j)|D_j| \to \iint_D f \, dA, \text{ where}$$

$\mathcal{A}$ consists of all D_j sitting inside D.

- Proof of the Change variables formula.

Consider

$$\Phi : G \longrightarrow D \quad \text{1-1, onto } C^1\text{-map.}$$



All $D_{jk} = \Phi(R_{jk})$, form a generalized partition on $\Phi(R)$, in particular on D.

Recall $\quad \Phi(u,v) = (g(u,v),\ h(u,v))$.

By Principle II, $\displaystyle\iint_D f\,dA$ can be approximated by

$$\sum_{\mathcal{A}} f(P_{jk})\,|D_{jk}|, \qquad \mathcal{A} \text{ consists of } D_{jk} \subset D.$$

As Φ is 1-1 onto, we can find $q_{jk} \in R_{jk}$ s.t. $\Phi(q_{jk}) = P_{jk}$.

$$\sum_{\mathcal{A}} f(P_{jk})\,|D_{jk}|$$

$$= \sum f(\Phi(q_{jk}))\,|D_{jk}|$$

$$= \sum \hat{f}(q_{jk}) \frac{|D_{jk}|}{|R_{jk}|}\,|R_{jk}|$$

$$= \sum \hat{f}(q_{jk}) \frac{|D_{jk}|}{|R_{jk}|}\,\Delta x_j \Delta y_k \,,$$ where the summation is

over all $R_{jk} \subset G$,

We'll show that

$$\frac{|D_{jk}|}{|R_{jk}|} \to \left|\frac{\partial(x,y)}{\partial(u,v)}\right|(u_{j-1}, v_{k-1}) \quad \text{as } \|P\| \to 0. \qquad (\bigstar)$$

then

$$\sum \hat{f}(q_{jk}) \frac{|D_{jk}|}{|R_{jk}|} \Delta x_j \Delta y_k$$

$$\approx \sum \hat{f}(q_{jk}) \left|\frac{\partial(x,y)}{\partial(u,v)}\right|(u_{j-1}, v_{k-1}) \Delta x_j \Delta y_k.$$

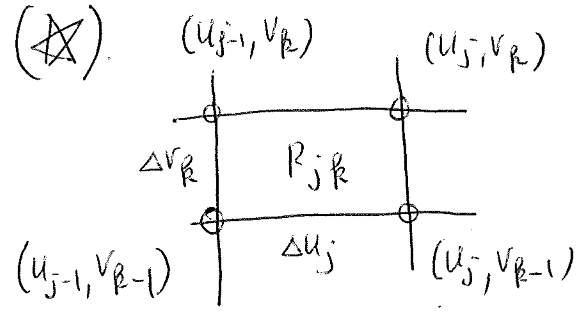
Since tag pts are arbitrary, take $q_{jk} = (u_{j-1}, v_{k-1})$,

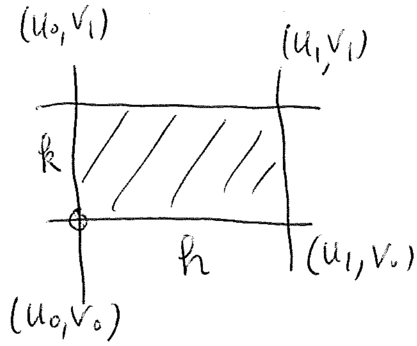
$$= \sum \hat{f}(u_{j-1}, v_{k-1}) \left|\frac{\partial(x,y)}{\partial(u,v)}\right|(u_{j-1}, v_{k-1}) \Delta x_j \Delta y_k$$

$$\to \iint_G \hat{f}(u,v) \left|\frac{\partial(x,y)}{\partial(u,v)}\right|(u,v)\, dA(u,v),$$

and the formula comes out.

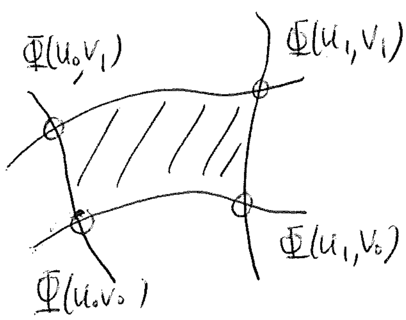
Pf of $(\bigstar)$





change notation.

We approximate the quadrilateral by a parallelogram:



$$\Phi(u_0, v_0) = (g(u_0, v_0), h(u_0, v_0))$$
$$\Phi(u_1, v_0) = \Phi(u_0 + h, v_0) = (g(u_0+h, v_0), h(u_0+h, v_0))$$
$$\Phi(u_0, v_1) = (g(u_0, v_0+k), h(u_0, v_0+k))$$
$$\Phi(u_1, v_1) = (g(u_0+h, v_0+k), h(u_0+h, v_0+k))$$

Use Taylor's thm: f C^2-function

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(z)(x - x_0)^2$$ where

z is between x_0 and x.

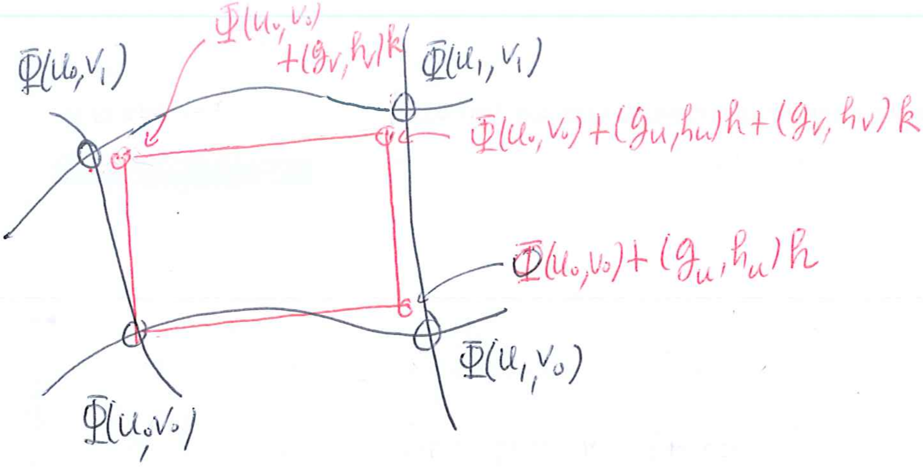
Appling to Φ,

$$g(u_0+h, v_0) = g(u_0, v_0) + g_u(u_0, v_0)\, h + \frac{1}{2} g_{uu}(z)\, h^2,$$
$$h(u_0+h, v_0) = h(u_0, v_0) + h_u(u_0, v_0)\, h + \frac{1}{2} h_{uu}(z)\, h^2,$$
$$g(u_0, v_1) = g(u_0, v_0) + g_v(u_0, v_0)\, k + \frac{1}{2} g_{vv}(z)\, k^2,$$
$$h(u_0, v_1) = h(u_0, v_0) + h_v(u_0, v_0)\, k + \frac{1}{2} h_{vv}(z)\, k^2,$$
$$g(u_1, v_1) = g(u_0, v_0) + g_u(u_0, v_0)\, h + g_v(u_0, v_0)\, k + \frac{1}{2}\left(g_{uu}(u_0, v_0)\, h^2 + 2 g_{uv}(u_0, v_0)\, hk + g_{vv}(u_0, v_0)\, k^2 \right),$$

$$h(u_1, v_1) = h(u_0, v_0) + h_u(u_0, v_0)\, h + h_v(u_0, v_0)\, k + \frac{1}{2}\left(h_{uu}(u_0, v_0)\, h^2 + 2 h_{uv}(u_0, v_0)\, hk + h_{vv}(u_0, v_0)\, k^2 \right).$$

(2-dim version of Taylor's thm see 2010)

Ignoring terms involving h^2, hk, k^2, we see that the quadrilateral with vertices at $\Phi(u_0, v_0)$, $\Phi(u_1, v_0)$, $\Phi(u_0, v_1)$, $\Phi(u_1, v_1)$ can be approximated by the parallelogram with vertices at $\Phi(u_0, v_0)$, $\Phi(u_0, v_0) + (g_u(u_0, v_0), h_u(u_0, v_0))\, h$, $\Phi(u_0, v_0) + (g_v(u_0, v_0), h_v(u_0, v_0))\, k$, and $\Phi(u_0, v_0) + (g_u(u_0, v_0), h_u(u_0, v_0))\, h + (g_v(u_0, v_0), h_v(u_0, v_0))\, k$.



The area of this parallolgram (the red one)

is $\left|(g_u,h_u)h \times (g_v,h_v)k\right|$

$$= \left|(g_u h_v - g_v h_u)\right| hk.$$

Therefore, as $\|P\| \to 0$ ie $h,k \to 0$,

$$\frac{|D_{jk}|}{|R_{jk}|} \cong \frac{\text{area of the parallegram}}{|R_{jk}|}$$

$$= \frac{\left|(g_u h_v - g_v h_u)(u_{j-1}, v_{k-1})\right| hk}{hk}$$

$$= \left|(g_u h_v - g_v h_u)(u_{j-1}, v_{k-1})\right|$$

$$= \left|\frac{\partial(x,y)}{\partial(u,v)}(u_{j-1}, v_{k-1})\right|, \ (\star) \text{ holds.}$$

This topic is optional and would not be tested. But important in the development of advanced calculus, try to understand it.